RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 5 Fall 2023

1. Christoffel Symbols for a Diagonal Metric

(a) From the definition, we have

$$\Gamma^{\bar\lambda}_{\bar\mu\bar\nu} = \frac{1}{2} g^{\bar\lambda\rho}(\partial_{\bar\mu} g_{\bar\nu\rho} + \partial_{\bar\nu} g_{\rho\bar\mu} - \partial_\rho g_{\bar\mu\bar\nu})\,. \tag{1}$$

If the metric is diagonal $g_{\bar\mu\bar\nu} = 0$ and the only non-zero element of $g^{\bar\lambda\rho}$ is $g^{\bar\lambda\bar\lambda}$, fixing $\rho = \bar\lambda$.

$$\Gamma^{\bar\lambda}_{\bar\mu\bar\nu} = \frac{1}{2} g^{\bar\lambda\bar\lambda}(\partial_{\bar\mu} g_{\bar\nu\bar\lambda} + \partial_{\bar\nu} g_{\bar\lambda\bar\mu}) = 0\,. \tag{2}$$

(b) As before, the only non-zero element is $\rho = \bar\lambda$

$$\Gamma^{\bar\lambda}_{\bar\mu\bar\mu} = \frac{1}{2} g^{\bar\lambda\bar\lambda}(\partial_{\bar\mu} g_{\bar\mu\bar\lambda} + \partial_{\bar\mu} g_{\bar\lambda\bar\mu} - \partial_{\bar\lambda} g_{\bar\mu\bar\mu}) = -\frac{1}{2g_{\bar\lambda\bar\lambda}}\partial_{\bar\lambda} g_{\bar\mu\bar\mu}\,. \tag{3}$$

(c) Same reasoning here

$$\Gamma^{\bar\lambda}_{\bar\mu\bar\lambda} = \frac{1}{2} g^{\bar\lambda\bar\lambda}(\partial_{\bar\mu} g_{\bar\lambda\bar\lambda} + \partial_{\bar\lambda} g_{\bar\lambda\bar\mu} - \partial_{\bar\lambda} g_{\bar\mu\bar\lambda}) = \frac{1}{2g_{\bar\lambda\bar\lambda}}\partial_{\bar\mu} g_{\bar\lambda\bar\lambda} = \partial_{\bar\mu}\left(\ln\sqrt{|g_{\bar\lambda\bar\lambda}|}\right)\,. \tag{4}$$

(d)

$$\Gamma^{\bar\lambda}_{\bar\lambda\bar\lambda} = \frac{1}{2} g^{\bar\lambda\bar\lambda}(\partial_{\bar\lambda} g_{\bar\lambda\bar\lambda} + \partial_{\bar\lambda} g_{\bar\lambda\bar\lambda} - \partial_{\bar\lambda} g_{\bar\lambda\bar\lambda}) = \frac{1}{2g_{\bar\lambda\bar\lambda}}\partial_{\bar\lambda} g_{\bar\lambda\bar\lambda} = \partial_{\bar\lambda}\left(\ln\sqrt{|g_{\bar\lambda\bar\lambda}|}\right)\,. \tag{5}$$

2. Geodesics

(a) The geodesic equations with affine parameter λ read

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0\,. \tag{6}$$

Making a generic change of variable $\lambda \to \alpha(\lambda)$ and using the chain rule, gives

$$\begin{aligned} \frac{d\alpha}{d\lambda}\frac{d}{d\alpha}\left(\frac{d\alpha}{d\lambda}\frac{dx^\mu}{d\alpha}\right) + \Gamma^\mu_{\rho\sigma}\left(\frac{d\alpha}{d\lambda}\right)^2\frac{dx^\rho}{d\alpha}\frac{dx^\sigma}{d\alpha} &= 0 \\ \left(\frac{d\alpha}{d\lambda}\right)^2\frac{d^2x^\mu}{d\alpha^2} + \frac{d^2\alpha}{d\lambda^2}\frac{dx^\mu}{d\alpha} + \Gamma^\mu_{\rho\sigma}\left(\frac{d\alpha}{d\lambda}\right)^2\frac{dx^\rho}{d\alpha}\frac{dx^\sigma}{d\alpha} &= 0 \\ \frac{d^2x^\mu}{d\alpha^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\alpha}\frac{dx^\sigma}{d\alpha} &= -\left(\frac{d\alpha}{d\lambda}\right)^{-2}\frac{d^2\alpha}{d\lambda^2}\frac{dx^\mu}{d\alpha}\,. \end{aligned} \tag{7}$$

We have thus found that

$$f(\alpha) = -\left(\frac{d\alpha}{d\lambda}\right)^{-2}\frac{d^2\alpha}{d\lambda^2}\,. \tag{8}$$

(b) The vector tangent to the geodesic $x^\mu(\lambda)$ has components

$$V^\mu = \frac{dx^\mu}{d\lambda}\,. \tag{9}$$

We can thus write

$$\begin{aligned} V^\mu \nabla_\mu V^\nu &= 0 \\ \frac{dx^\mu}{d\lambda}\left(\partial_\mu \frac{dx^\nu}{d\lambda} + \Gamma^\nu_{\mu\rho}\frac{dx^\rho}{d\lambda}\right) &= 0 \\ \frac{dx^\mu}{d\lambda}\partial_\mu \frac{dx^\nu}{d\lambda} + \Gamma^\nu_{\mu\rho}\frac{dx^\mu}{d\lambda}\frac{dx^\rho}{d\lambda} &= 0 \\ \frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\rho}\frac{dx^\mu}{d\lambda}\frac{dx^\rho}{d\lambda} &= 0\,, \end{aligned} \tag{10}$$

which is the geodesic equation.

3. Geodesics on S^2

(a) To write down the geodesic equations explicitly, we need to first compute the Christoffel symbols. This is a diagonal metric, so we can use the formulae derived in Problem 1. We find that the only non-vanishing Christoffel symbols are

$$\begin{aligned} \Gamma^\theta_{\phi\phi} &= -\frac{1}{2g_{\theta\theta}}\partial_\theta g_{\phi\phi} = -\frac{1}{2}\partial_\theta \sin^2\theta = -\sin\theta\cos\theta \\ \Gamma^\phi_{\phi\theta} &= \partial_\theta\left(\ln\sqrt{|g_{\phi\phi}|}\right) = \partial_\theta\left(\ln\sqrt{|\sin^2\theta|}\right) = \frac{\cos\theta}{\sin\theta} = \Gamma^\phi_{\theta\phi}\,. \end{aligned} \tag{11}$$

where we used the fact that the metric elements in these coordinates only depend on θ, so derivatives with respect to ϕ immediately vanish. The geodesic equation has a free index, which we can choose to be θ or ϕ. We thus get two equations

$$\begin{cases} \frac{d^2x^\theta}{d\lambda^2} + \Gamma^\theta_{\mu\rho}\frac{dx^\mu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \\ \frac{d^2x^\phi}{d\lambda^2} + \Gamma^\phi_{\mu\rho}\frac{dx^\mu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \end{cases} \tag{12}$$

The non-vanishing terms are thus

$$\begin{cases} \frac{d^2x^\theta}{d\lambda^2} + \Gamma^\theta_{\phi\phi}\left(\frac{dx^\phi}{d\lambda}\right)^2 = 0 \\ \frac{d^2x^\phi}{d\lambda^2} + 2\Gamma^\phi_{\theta\phi}\frac{dx^\theta}{d\lambda}\frac{dx^\phi}{d\lambda} = 0 \end{cases}\,. \tag{13}$$

Defining $x^\theta(\lambda) \equiv \theta(\lambda)$ and $x^\phi(\lambda) \equiv \phi(\lambda)$ and substituting the expressions of the Christoffels that we computed, we get

$$\begin{cases} \ddot\theta - \sin\theta\cos\theta\dot\phi^2 = 0 \\ \ddot\phi + 2\cot\theta\;\dot\theta\dot\phi = 0 \end{cases}\,, \tag{14}$$

where dots represent derivatives with respect to λ.

(b) If we impose constant longitude $\phi = \phi_0$ the geodesic equations reduce to

$$\ddot{\theta} = 0\,, \tag{15}$$

meaning that any curve $\theta(\lambda) = u\lambda + \theta_0$ with u and θ_0 constants is a good geodesic.If instead we impose $\theta = \theta_0$ the equation becomes

$$\begin{cases} \sin\theta_0 \cos\theta_0 \dot{\phi}^2 = 0 \\ \ddot{\phi} = 0 \end{cases} . \tag{16}$$

That means that $\theta = \theta_0$ has solutions for any curve $\phi(\lambda) = u\lambda + \phi_0$ only if

$$\theta_0 = 0, \frac{\pi}{2}, \pi\,, \tag{17}$$

corresponding to the poles and the equator. At the poles these geodesics would represent points, so the only real extended geodesic of constant latitude $\theta = \theta_0$ is the equator.

(c) A vector V is parallel transported along a curve $x^\nu(\lambda)$ when

$$\frac{dx^\mu}{d\lambda} \nabla_\mu V^\nu = 0\,. \tag{18}$$

Again this corresponds to two equations for the two choices of the free index ν. We want to parallel transport this vector on a curve of constant longitude, meaning $\theta = \theta_0$. The equations become

$$\begin{cases} \dot{\phi}\left(\frac{\partial V^\theta}{\partial \phi} + \Gamma^\theta_{\phi\phi} V^\phi\right) = 0 \\ \dot{\phi}\left(\frac{\partial V^\phi}{\partial \phi} + \Gamma^\phi_{\phi\theta} V^\theta\right) = 0\,, \end{cases} \tag{19}$$

where we used that derivatives of θ vanish. Since we want to consider general $\phi(\lambda)$, the differential equations on the components of V reduce to

$$\begin{cases} \frac{\partial V^\theta}{\partial \phi} - \sin\theta_0 \cos\theta_0 V^\phi = 0\,, \\ \frac{\partial V^\phi}{\partial \phi} + \cot\theta_0 V^\theta = 0 \end{cases} . \tag{20}$$

Given that the initial conditions for the vector components are $V^\mu = (1, 0)$ we get initial conditions for its derivatives

$$\begin{cases} \left.\frac{\partial V^\theta}{\partial \phi}\right|_{\phi=0} = 0\,, \\ \left.\frac{\partial V^\phi}{\partial \phi}\right|_{\phi=0} + \cot\theta_0 = 0 \end{cases} . \tag{21}$$

At this point we could put (20) on Mathematica and find the solution. Another option is to find decoupled equations by, for example, deriving the first equation with respect to ϕ and substituting $\frac{\partial V^\phi}{\partial \phi}$ from the second one and vice versa. We get

$$\begin{cases} \frac{\partial^2 V^\theta}{\partial \phi^2} + \cos^2\theta_0 V^\theta = 0\,, \\ \frac{\partial^2 V^\phi}{\partial \phi^2} + \cos^2\theta_0 V^\phi = 0 \end{cases} . \tag{22}$$

These are decoupled harmonic oscillators with frequency $\cos\theta_0$. General solutions are

$$\begin{cases} V^\theta(\phi) = A\cos(\phi\cos\theta_0) + B\sin(\phi\cos\theta_0)\,, \\ V^\phi(\phi) = C\cos(\phi\cos\theta_0) + D\sin(\phi\cos\theta_0)\,. \end{cases} \tag{23}$$

Imposing the initial conditions of V^μ and its derivatives, we get

$$\begin{cases} V^\theta(\phi) = \cos(\phi\cos\theta_0)\,, \\ V^\phi(\phi) = -\frac{\sin(\phi\cos\theta_0)}{\sin\theta_0}\,. \end{cases} \tag{24}$$

Thus, parallel transporting a vector with initial components $(1,0)$ around a circle of constant latitude $\theta = \theta_0$ gives a vector of components

$$\begin{cases} V^\theta(2\pi) = \cos(2\pi\cos\theta_0)\,, \\ V^\phi(2\pi) = -\frac{\sin(2\pi\cos\theta_0)}{\sin\theta_0}\,. \end{cases} \tag{25}$$

We can verify again that $\theta_0 = \frac{\pi}{2}$ is a geodesic: when we plug in this value, we get

$$\begin{cases} V^\theta(2\pi) = 1\,, \\ V^\phi(2\pi) = 0\,, \end{cases} \tag{26}$$

as it should, since a vector that is parallel transported along a geodesic on a closed circuit preserves its components.

4. Divergence and Laplacian

(a) Let us start with the explicit expression of the covariant derivative

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\nu} V^\nu\,, \tag{27}$$

where by definition

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\alpha}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right) = \frac{1}{2} g^{\mu\alpha}\partial_\nu g_{\alpha\mu}\,. \tag{28}$$

This, in terms of matrices is

$$\frac{1}{2}\mathrm{Tr}\left(\mathbf{g}^{-1}\partial_\nu \mathbf{g}\right) \tag{29}$$

From the identity

$$\mathrm{Tr}\left(\mathbf{M}^{-1}\partial_\nu \mathbf{M}\right) = \partial_\nu\left(\ln\det\mathbf{M}\right), \tag{30}$$

coming from differentiating the common identity $\log\det\mathbf{M} = \mathrm{Tr}\log\mathbf{M}$, one gets

$$\frac{1}{2} g^{\mu\alpha}\partial_\nu g_{\alpha\mu} = \frac{1}{\sqrt{|g|}}\partial_\nu\sqrt{|g|} \tag{31}$$

bringing the expression to the desired form

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}\right)V^\nu = \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}V^\nu\right)\,. \tag{32}$$

(b) The identity for the Laplacian trivially follows from setting $V^\mu = \nabla^\mu f = g^{\mu\nu}\partial_\nu f$